Topic: ∘ Integrability of continuous function over compact set

∘ Uniform Continuity V.S. Pointwise Continuity

Notation: - $Q = [a,b] \times [c,d] \subset \mathbb{R}^2$

- Area of $Q = (b-a)(d-c)$
- Let $f: Q \to \mathbb{R}$ be a bounded function

i.e. $\exists\, m, M \in \mathbb{R}$ s.t. $m \le f(x) \le M \quad \forall x \in Q$.

- Let $\begin{cases} a = x_0 < x_1 < \cdots < x_n = b \\ c = y_0 < y_1 < \cdots < y_m = d \end{cases}$, $m, n \in \mathbb{N}$.

The collection $\{[x_i, x_{i+1}] \times [y_j, y_{j+1}] \mid i = 0, \cdots, n-1, j = 0, \cdots, m-1\}$ is called a partition of Q.

- A function $s: Q \to \mathbb{R}$ is called a step function if $\exists$ a partition $\{[x_i, x_{i+1}] \times [y_j, y_{j+1}] \mid i, j\}$ on Q s.t.

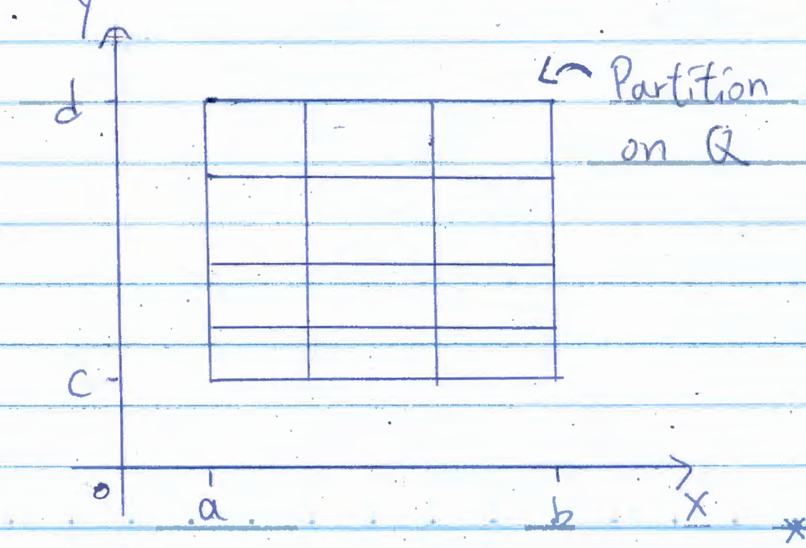
$$s\Big|_{[x_i, x_{i+1}] \times [y_j, y_{j+1}]} = \underset{c_{ij}}{\text{constant}}$$

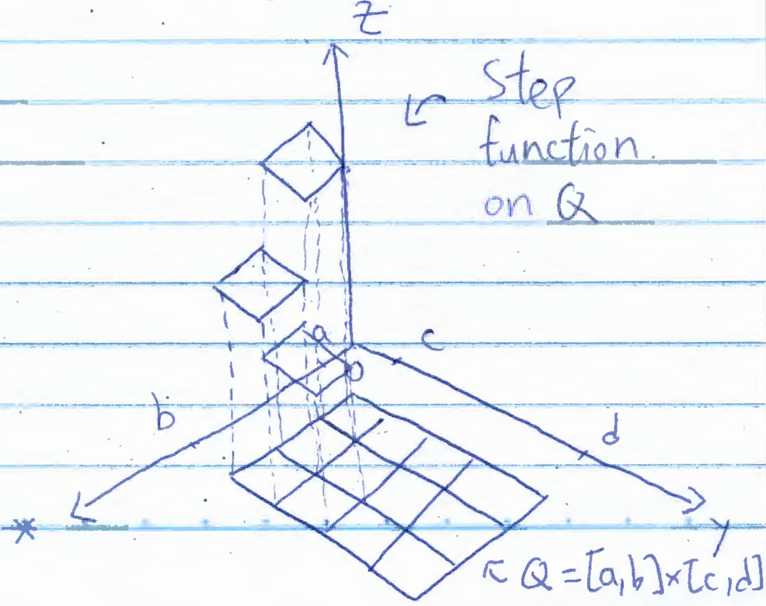
Defn: Let $s: Q \to \mathbb{R}$ be a step function.

The integral of s over Q is defined by

$$\iint_Q s = \iint_Q s(x,y)\,dx\,dy = \sum_{i,j} c_{ij}\,\Delta x_i\,\Delta y_j,$$

where $\Delta x_i = x_{i+1} - x_i$, $\Delta y_j = y_{j+1} - y_j$.





Defn: ∘ The <u>upper integral</u> of $f: Q \to \mathbb{R}$ is defined by

$$\overline{\iint_Q} f = \overline{\iint_Q} f(x,y)\,dx\,dy = \inf \left\{ \iint_Q t \;\middle|\; \begin{array}{l} t = \text{step function on } Q \\ \text{s.t. } f \le t \end{array} \right\}$$

∘ The <u>lower integral</u> of $f: Q \to \mathbb{R}$ is defined by

$$\underline{\iint_Q} f = \underline{\iint_Q} f(x,y)\,dx\,dy = \sup \left\{ \iint_Q s \;\middle|\; \begin{array}{l} s = \text{step function on } Q \\ \text{s.t. } s \le f \end{array} \right\}$$

Defn: f is <u>integrable</u> on Q if $\underline{\iint_Q} f = \overline{\iint_Q} f$.

Thm: Every continuous function $f: Q \to \mathbb{R}$ is integrable.

Pf: "Uniform Continuity $\Rightarrow$ Integrable"

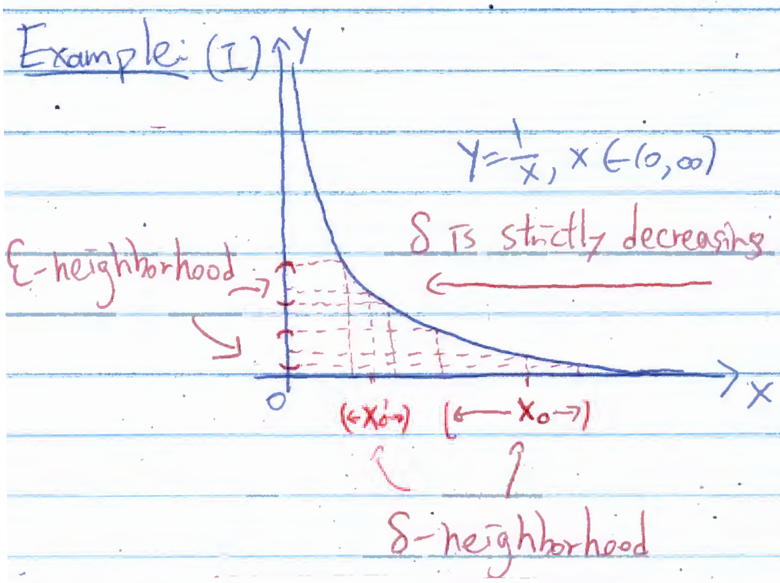
<u>Uniform Continuity V.S. Pointwise Continuity:</u>

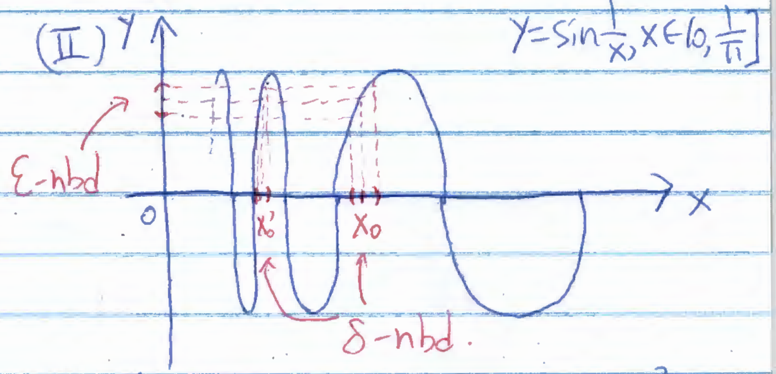
Defn: (Pointwise Continuous)

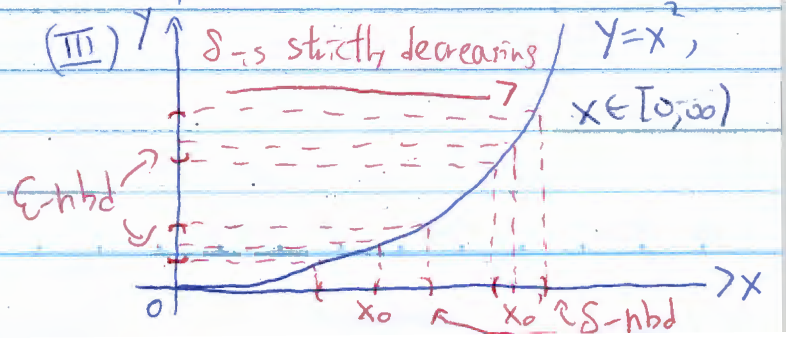
- A function $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous at $x_0 \in D$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta$.

depends on x_0.

Example:



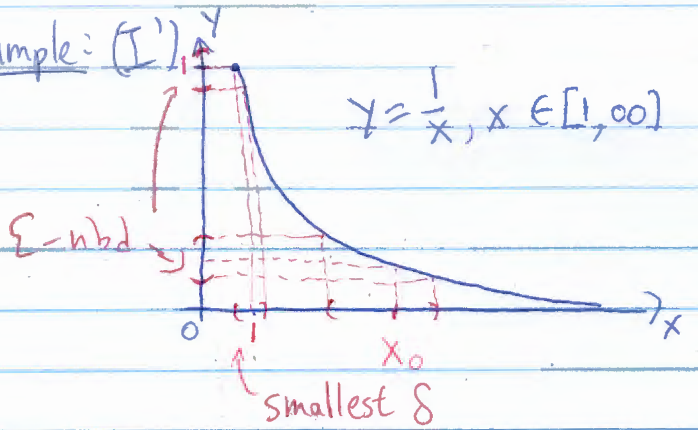




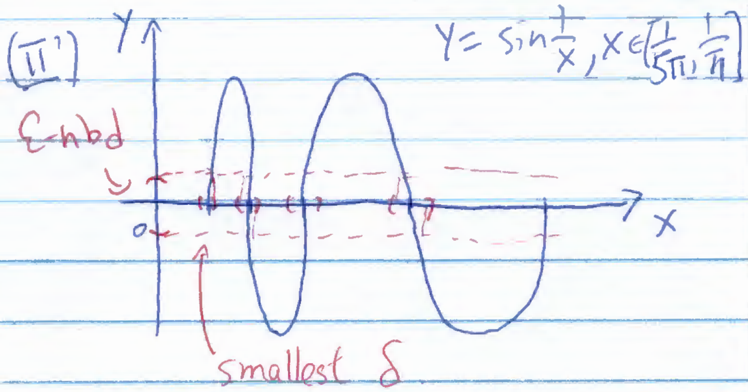
Defn: (Uniform Continuous)

- A function $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is uniform continuos if $\forall \varepsilon > 0, \exists \delta > 0$ (not depend on x, y) s.t. $|f(x) - f(y)| < \varepsilon$ whenever $\|x-y\| < \delta$

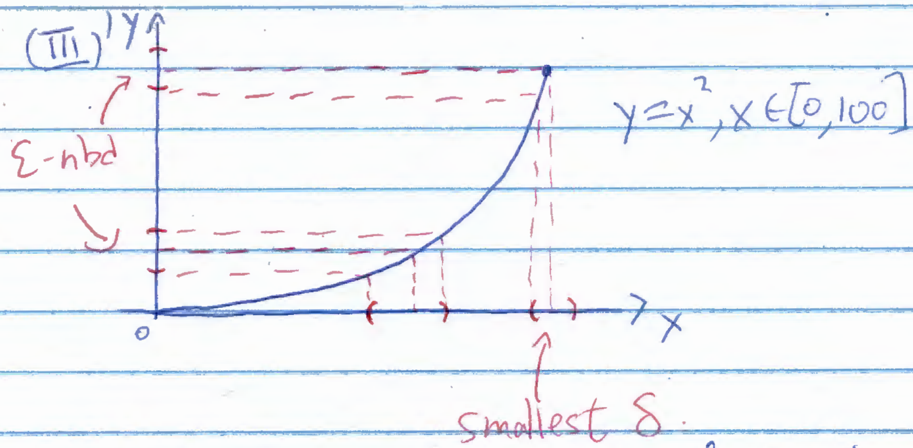
Example: (I')



(II')



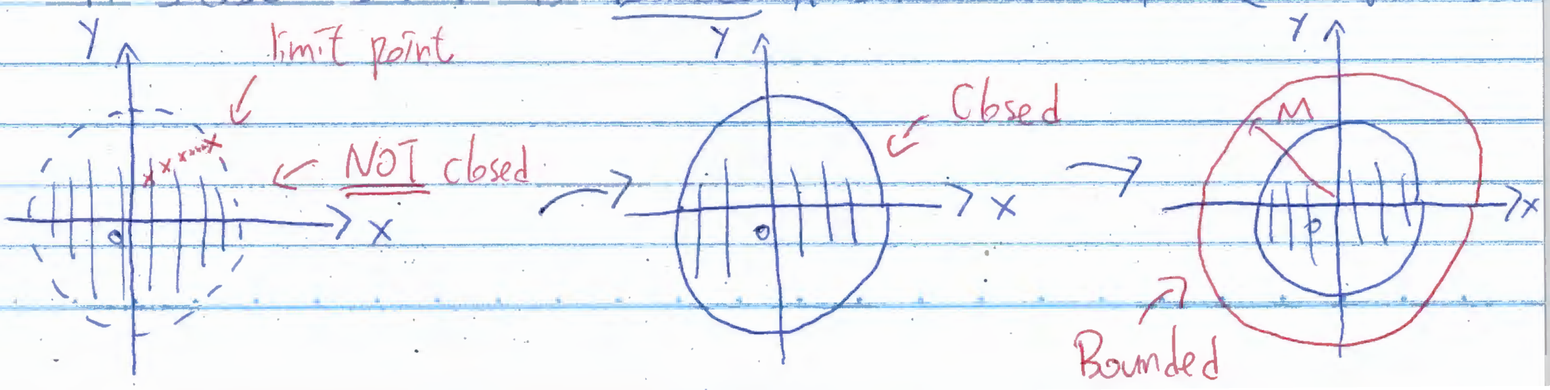
(III')



Remark: Uniform continuity depends on the property of the domain.

Defn: (Limit Point, Closed set, Bounded set)

- A point $x_0 \in \mathbb{R}^2$ is a limit point of $D \subseteq \mathbb{R}^2$ if
- $\exists$ a sequence $\{x_n\} \subseteq D$ s.t. $x_n \to x_0$.
  i.e. $\|x_n - x_0\| \to 0$ as $n \to \infty$.
- A subset $D \subseteq \mathbb{R}^2$ is closed if it contains all its limit points.
- A subset $D \subseteq \mathbb{R}^2$ is bounded if $\exists M > 0$ s.t. $\|x\| \le M \ \forall x \in D$.



(Roughly speaking, a subset D is closed if it contains its boundary)

Defn: (Compact set)
- A subset $D \subseteq \mathbb{R}^2$ is compact if it is closed and bounded.

Now, we are going to define integration of f over compact set.

Defn: Let $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function over compact set D.
Choose a rectangle Q s.t. $Q \supseteq D$.
Define $\tilde{f}: Q \to \mathbb{R}$ by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in Q \setminus D \end{cases}$$

Define the integral of f over D by

$$\iint_D f = \iint_Q \tilde{f}$$

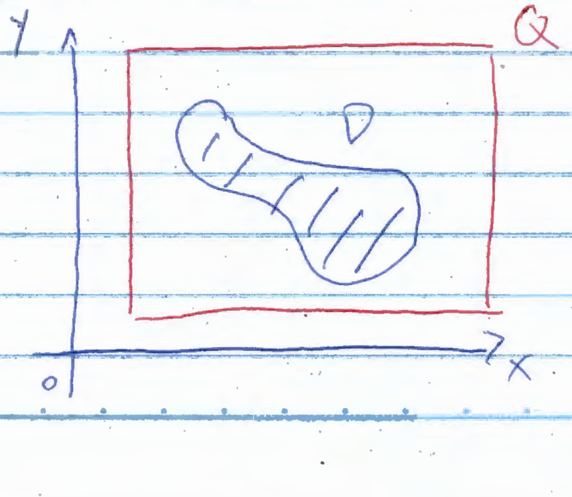
Thm: Every continuous function $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ on a compact set D is uniform continuous.

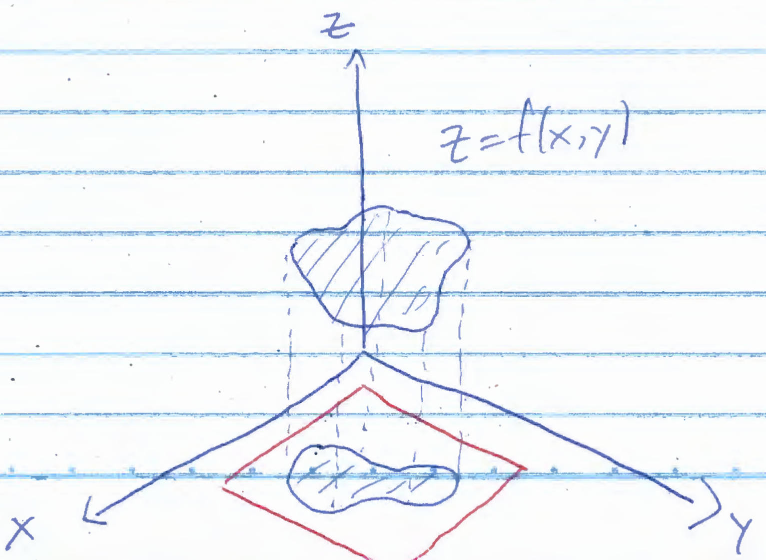
Pf: Skip. The proof requires some analysis.

Assuming the theorem above, we have the following theorem:

Thm: Every continuous function $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ on a compact set D is integrable.

Pf: (Sketch)

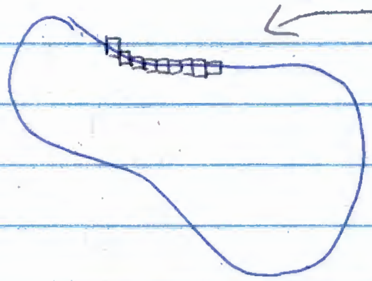




① Outside D, as $\tilde{f}$ is zero over there, we may choose the values of the step functions to be zero.

② Inside D, as $\tilde{f}=f$ is uniform continuous, the integral of the steps function can be arbitarily close.

③ On the boundary of D, as it can be covered by small rectangles, it contributes nothing to the integral.



← Sum of area of rectangles can be arbitrary small.

$\Rightarrow$ Sum of volume of cuboid can be arbitrary small.

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Remark: For more detail information about ③, please refer to "the supplementary note on tutorial 2".